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TEMPERATURE DISTRIBUTION ON CONDUCTING CYLINDRICAL SHELLS INCLUDING THE EFFECTS OF THERMAL RADIATION*

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1. Introduction. In the near-vacuum environment of space vehicles, heat transfer to or from a given structure is predominantly a radiative phenomenon. For relatively few cases, however, is the structural geometry such that predictions can be achieved in a general form. The present paper deals with an idealization of practical interest for which theoretical predictions can, in fact, be given explicitly. In the case under study, the dependence of the heat transfer upon parametric values of structural emissivity, absorptivity, and thermal conductivity is thus exhibited analytically.

We shall be concerned with a thin, conducting cylindrical shell, either closed or open but of sufficient length that a two-dimensional idealization is possible and heat transfer is calculated in a cross section. An external source of radiation is assumed. The thermal conductivity of the shell is known and the radiative emission from the inner and outer surfaces is taken to be diffuse; also, no dependence on radiation wavelength is employed and a gray body type analysis is thus used.

Success in treating this problem becomes essentially a matter of solving an integro-differential equation. Two further restrictions make the solution feasible: first, temperature ranges should be sufficiently restricted that perturbation methods apply; second, the radiative influence function that is the kernel of the integral term should be amenable to analytic treatment. The first restriction is made as an initial assumption; its validity for particular values of the parameters can be tested subsequently. The second restriction leads naturally to the consideration of circular arc sections.

The resulting integro-differential equation has appeared in more or less restrictive form in the literature. In a classical paper Jensen [1] studied the heat transfer between two bodies of different temperature, one of them entirely surrounding the other, and was led to an integral equation that was formally similar. More recently, Frank and Gray [2] and Parkes [3] treated simpler cases of the present physical problem. In each of these papers an eigenfunction type analysis was used which resulted in infinite series solutions following quadrature of the radiation source function. The present results extend the generality and show that the series can be summed. The

method of attack is a natural extension of the technique employed by Heaslet and Lomax [4] to study nonconducting structures, namely, the reduction of the governing equation to a higher order differential equation. Here, and in [4], inversions are given that are apparently novel. E. M. Sparrow [5] has recently studied radiating, nonconducting cylindrical surfaces by analytical methods similar to those employed in [4]. Sparrow's work, which came to the author's attention during the editorial review of the present paper, treats both diffusely and specularly reflecting surfaces.

Important symbols to be used are listed below:

a radius of cylinder cross section (Fig. 1);

$B(\theta)$ energy flux (cal/cm^{2}sec) leaving a surface;

$f(\theta)$ incident radiation (cal/cm^{2}sec), normal to surface, from external source;

$F(\theta)$ $f(\theta)/\sigma T_0^4$;

$H(\theta)$ energy flux (cal/cm^{2}sec) impinging on a surface;

k thermal conductivity $\left[\left(\dfrac{\text{cal}}{\text{sec cm}^2} \right) \left(\dfrac{{}^\circ\text{K}}{\text{cm}} \right)^{-1} \right]$;

N $a^2 \sigma T_0^3 / kt$;

t thickness of cylinder wall (Fig. 1);

T temperature;

T_0 average temperature, see (6b);

u $\dfrac{T}{T_0} - 1$;

α_1 absorptivity, outer surface of cylinder;

α_2 absorptivity, inner surface of cylinder;

β see (10a);

$\Delta(f, g)$ see after (15);

ϵ_1 emissivity, outer surface of cylinder;

ϵ_2 emissivity, inner surface of cylinder;

ν see (10a);

σ Stefan-Boltzman constant;

2ψ angular extent of cross section of cylinder (Fig. 1).

2. The governing equation. We take the basic configuration to be a long shell with cross section specified as a circular arc. Fig. 1 shows a section normal to the axis of the cylinder. It is assumed at the outset that differences can exist between the constant absorptivity α and emissivity ϵ of a surface, thus allowing for the possibility of a high temperature radiation source and a surface emissivity corresponding to a much lower temperature. Subscripts 1 and 2 are used to denote conditions on the outer and inner surfaces, respectively. The thermal conductivity k of the material is assumed constant throughout. We assume $t/a \ll 1$ and proceed to the
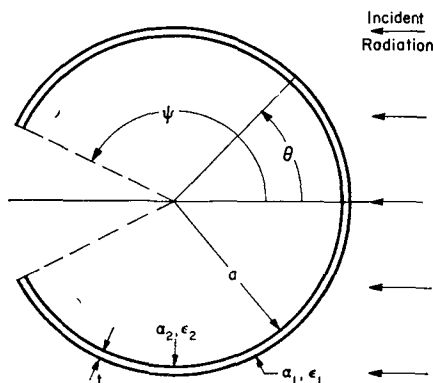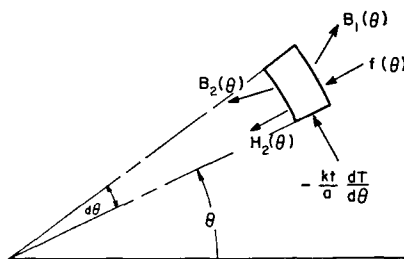
FIG. 1.

FIG. 2.

development of a thin shell theory in which the gradient of temperature through the shell is neglected. The angular coordinate θ is measured from the axis of symmetry of the arc, positive values corresponding to a counterclockwise rotation. Thus over the full extent of the shell θ covers the range from $-\psi$ to ψ.

The governing equations are derived through consideration of the element of area shown in Fig. 2. For unit depth normal to the plane of the element, the following balances occur:

external face

(1a) $$B_1(\theta) = \epsilon_1 \sigma T^4(\theta) + (1 - \alpha_1)f(\theta);$$

internal face

(1b) $$B_2(\theta) = \epsilon_2 \sigma T^4(\theta) + (1 - \alpha_2)H_2(\theta);$$

energy balance within skin

(1c) $$d\left(-\frac{kt}{a}\frac{dT}{d\theta}\right) + [B_1(\theta) - f(\theta)]\,a\,d\theta + [B_2(\theta) - H_2(\theta)]a\,d\theta = 0.$$

In (1a), $B_1(\theta)$ is the energy flux (cal/cm^{2}sec) of diffuse radiation from the external face. This flux is equal to the emissive power of unit area according to the Stefan-Boltzmann law, $\epsilon_1 \sigma T^4(\theta)$, plus the reflected portion of the incident radiation flux distribution, $(1 - \alpha_1)f(\theta)$, where $f(\theta)$ is measured in cal/cm^{2}sec. The reflectivity factor $\rho_1 = 1 - \alpha_1$ holds for opaque surfaces.

In (1b), $B_2(\theta)$ is the energy flux from the internal face. This flux is equated to the emissive power per unit area, $\epsilon_2 \sigma T^4(\theta)$, plus the reflected portion of the incoming flux from within the enclosure. The incoming flux

of radiation is denoted $H_2(\theta)$ and is given by

$$(2) \qquad H_2(\theta) = \int_{-\psi}^{\psi} B_2(\theta_1)\, dF_{\theta,\theta_1},$$

where the integral sums the contribution of emission from elements of arc along the inner face of the enclosure and dF_{θ,θ_1} is the conventional angle factor that measures the intensity at the position θ of the emission from the surface element at θ_1. Equation (2) implies that there is no radiation impinging on the cylinder from the left in Fig. 1. If it were desired to include such a feature, it would be necessary to add terms to (2) which would take it into account. This would necessitate some care being taken in the treatment of possible shadow zones in the interior portion of the partially open cylinder, but the nature of the analysis remains essentially unchanged. For a circular arc the two-dimensional angle factor for diffuse radiation takes the particularly simple form

$$dF_{\theta,\theta_1} = \frac{1}{4} \sin \left| \frac{\theta - \theta_1}{2} \right| d\theta_1,$$

where the bars denote absolute values of the argument. Equations (1b) and (2) thus combine to yield

$$(3) \qquad B_2(\theta) = \epsilon_2\, \sigma T^4(\theta) + \frac{1 - \alpha_2}{4} \int_{-\psi}^{\psi} B_2(\theta_1) \sin \left| \frac{\theta - \theta_1}{2} \right| d\theta_1.$$

Equation (1c) represents a balance between conduction and radiation. The first term is the difference between values at θ and $\theta + d\theta$ of the conductive energy transport, as given by Fourier's law, along the interior of the shell. The additional terms yield the energy per unit time transported by radiation away from elemental portions of the exposed surfaces.

Equation (1b) yields the relations

$$(4) \qquad B_2(\theta) - H_2(\theta) = \epsilon_2\, \sigma T^4(\theta) - \alpha_2 H_2(\theta) = \frac{\epsilon_2\, \sigma T^4(\theta) - \alpha_2 B_2(\theta)}{1 - \alpha_2}$$

and these, together with (1) and (3), give

$$(5) \quad \begin{aligned} &\left[\frac{kt}{a^2} \frac{d^2 T}{d\theta^2} - \left(\epsilon_1 + \frac{\epsilon_2}{1 - \alpha_2} \right) \sigma T^4(\theta) + \alpha_1 f(\theta) \right] \\ &\qquad = \frac{-\alpha_2\, \epsilon_2}{1 - \alpha_2} \sigma T^4(\theta) + \frac{1 - \alpha_2}{4} \int_{-\psi}^{\psi} \left[\quad \right] \sin \left| \frac{\theta - \theta_1}{2} \right| d\theta_1 \end{aligned}$$

where the expression within the brackets in the integrand is the expression that appears within brackets on the left side of the equation (with θ_1 replacing θ).

Equation (5) is the integro-differential expression that governs temperature distribution for a prescribed input of radiation $f(\theta)$. Nonlinear analysis is obviously required in general. However, for physical problems in which the magnitude of $T(\theta)$ has a restricted variation, linearization of the equation is indicated. To this end, we first introduce the dimensionless variable

$$(6a) \qquad u(\theta) = \frac{T}{T_0} - 1,$$

where T_0 is the average temperature of the cylindrical section:

$$(6b) \qquad \frac{1}{2\psi} \int_{-\psi}^{\psi} \frac{T(\theta)}{T_0} \, d\theta = \frac{1}{2\psi} \int_{-\psi}^{\psi} (1 + u) \, d\theta = 1.$$

Assume now that

$$(7) \qquad \left(\frac{T}{T_0}\right)^4 = 1 + 4u$$

is an adequate approximation. This can be examined a posteriori by inspection of the solution. Using the approximation of (7), one gets the linearized form of (5) as

$$
\begin{aligned}
(8) \quad & \left[\frac{d^2u}{d\theta^2} - 4N\left(\epsilon_1 + \frac{\epsilon_2}{1 - \alpha_2}\right) u + N\alpha_1 F(\theta)\right] \\
& \quad + 4N \frac{\alpha_2 \epsilon_2}{1 - \alpha_2} u - \frac{1 - \alpha_2}{4} \int_{-\psi}^{\psi} \left[\qquad \right] \sin \frac{1}{2} \mid \theta - \theta_1 \mid d\theta_1 \\
& \qquad = N\epsilon_1 \alpha_2 + N(1 - \alpha_2)\left(\epsilon_1 + \frac{\epsilon_2}{1 - \alpha_2}\right) \cos \frac{1}{2} \psi \cos \frac{1}{2} \theta,
\end{aligned}
$$

where $F(\theta) = f(\theta)/\sigma T_0^4$, $N = a^2 \sigma T_0^3/kt$ and, as previously, the terms within the brackets are functionally the same. The parameter N provides a measure of the relative strengths of radiation effects and thermal conduction within the shell. This is more apparent when it is written in the form $N = (a\sigma T_0^4)/(ktT_0/a)$.

Equation (8) is to be solved for $u(\theta)$ as a functional of $f(\theta)$. Two boundary conditions are needed since the equation contains a second-order differential. These conditions are obtained by imposing a radiation condition at the extremities of the shell. Then, at $\theta = \pm\psi$, $-\frac{kt}{a}\frac{dT}{d\theta} = \epsilon_2 \sigma T^4 t$. If $t/a \ll 1$, this becomes, using (6a) and (7),

$$(9) \qquad \frac{du}{d\theta} = -N\epsilon_2 \left(\frac{t}{a}\right)(1 + 4u) \doteq 0.$$

The necessary boundary conditions might also be obtained by prescribing the fixed temperatures T_1, T_2 at which the ends of the arc are to be held. In this case the conditions are

(9a)
$$u(\psi) = \frac{T_1}{T_0} - 1, \qquad u(-\psi) = \frac{T_2}{T_0} - 1.$$

The solution of (8) subject to the end conditions of (9) can be attained through a reduction of the integro-differential equation to a linear differential equation. Operating upon (8) with $(D^2 + \frac{1}{4})$ where $D \equiv d/d\theta$ we get the fourth-order equation

(10)
$$\left\{D^4 - \left[4N(\epsilon_1 + \epsilon_2) - \frac{\alpha_2}{4}\right]D^2 - N\alpha_2\epsilon_1\right\}u$$
$$= -N\alpha_1\left(D^2 + \frac{\alpha_2}{4}\right)G - \frac{1}{4}N\frac{\alpha_2\epsilon_2}{1-\alpha_2},$$

where $G(\theta) = F(\theta) - \frac{1}{\alpha_1}\left(\epsilon_1 + \frac{\epsilon_2}{1-\alpha_2}\right)$. The increase in order of the resulting differential equation introduces two additional arbitrary constants in the solution. No additional conditions are required, however, since the values of these constants will be fixed through substitution of the general expression for $u(\theta)$ into the governing equation (8).

3. General solution. In this section we give the solution for (8), together with the boundary conditions (9), in the case of arbitrary angle ψ (see Fig. 1). When this general solution has been determined, it will be specialized, in the following section, to the case of a complete cylinder, $\psi = \pi$, for which some numerical results will be given.

The first step will be to factor the quartic differential operator appearing on the left in (10). We write that equation as

(10a)
$$[(D^2 + \nu^2)(D^2 - \beta^2)]u(\theta)$$
$$= -N\alpha_1\left(D^2 + \frac{\alpha_2}{4}\right)G(\theta) - \frac{1}{4}N\frac{\alpha_2\epsilon_2}{1-\alpha_2},$$

where

$$\nu^2 = -\tfrac{1}{2}\{4N(\epsilon_1 + \epsilon_2) - \alpha_2/4 - ([4N(\epsilon_1 + \epsilon_2) - \alpha_2/4]^2 + 4\epsilon_1\alpha_2 N)^{1/2}\} > 0,$$
$$\beta^2 = \nu^2 + 4(\epsilon_1 + \epsilon_2)N - \alpha_2/4,$$
$$(\alpha_2/4 + \beta^2)(\alpha_2/4 - \nu^2) = N\alpha_2\epsilon_2,$$
$$\nu^2\beta^2 = N\epsilon_1\alpha_2.$$

The last two relations are useful in the algebraic reductions. Operate on (10a) with the inverse $[(D^2 + \nu^2)(D^2 - \beta^2)]^{-1}$; there results for the coefficient of $G(\theta)$

$$(11) \qquad -\frac{\alpha_1 N}{\nu^2 + \beta^2}\left(\frac{\nu^2 - \alpha_2/4}{D^2 + \nu^2} + \frac{\beta^2 + \alpha_2/4}{D^2 - \beta^2}\right).$$

Interpreting this operational form, we are led to the particular solution

$$
\begin{aligned}
(12) \qquad u_p(\theta) = {}& -\frac{\alpha_1 N}{\nu^2 + \beta^2}\left[\frac{\nu^2 - \alpha_2/4}{2\nu}\int_{-\psi}^{\psi} G(\theta_1)\sin\nu\,|\,\theta - \theta_1\,|\,d\theta_1\right. \\
& \left. + \frac{\beta^2 + \alpha_2/4}{2\beta}\int_{-\psi}^{\psi} G(\theta_1)\sinh\beta\,|\,\theta - \theta_1\,|\,d\theta_1\right] + \frac{\epsilon_2/\alpha_2}{4(1 - \alpha_2)}.
\end{aligned}
$$

To this particular solution we must expect to add complementary solutions

$$(13) \qquad u_c(\theta) = A_1\cos\nu\theta + B_1\sin\nu\theta + A_2\cosh\beta\theta + B_2\sinh\beta\theta,$$

so that the complete solution of (10a) is

$$(14) \qquad u(\theta) = u_p(\theta) + u_c(\theta).$$

The arbitrary constants A_1, A_2, B_1, B_2 are to be determined by resubstitution into (8) and by use of the boundary conditions at $\theta = \pm\psi$ given by (9). Further, the value of T_0 can be determined by an application of (6b) to the final form of the solution.

First, let us note the result of substituting the value of $u(\theta)$ given by (14) back into the integro-differential equation (8). In the present notation, the latter equation is

$$
\begin{aligned}
(15) \qquad & (D^2 + \nu^2 - \beta^2 - \alpha_2/4)u(\theta) \\
& - \frac{1 - \alpha_2}{4}\int_{-\psi}^{\psi}\left(D^2 - \frac{\beta^2 - \nu^2 + \alpha_2/4 - 4\nu^2\beta^2}{1 - \alpha_2}\right)u(\theta_1)\sin\frac{1}{2}\,|\,\theta - \theta_1\,|\,d\theta_1 \\
& = -N\alpha_1\left[G(\theta) - \frac{1 - \alpha_2}{4}\int_{-\psi}^{\psi} G(\theta_1)\sin\frac{1}{2}\,|\,\theta - \theta_1\,|\,d\theta_1\right] - N\frac{\alpha_2\,\epsilon_2}{1 - \alpha_2}.
\end{aligned}
$$

The result of the substitution is a pair of linear equations involving the arbitrary constants A_i, B_i of (13). It is convenient to introduce the notation $\Delta(f(\psi), g(\psi)) = \begin{vmatrix} f & g \\ f' & g' \end{vmatrix}$, where f, g are functions of ψ, and the prime denotes differentiation with respect to ψ. The two linear equations are

$$2 \frac{\Delta(\cos \nu\psi, \sin \frac{1}{2}\psi)}{\alpha_2/4 - \nu^2} A_1 + 2 \frac{\Delta(\cosh \beta\psi, \sin \frac{1}{2}\psi)}{\alpha_2/4 + \beta^2} A_2$$

(16a)

$$= -\frac{1}{\alpha_2 \epsilon_1}\left(\epsilon_1 + \frac{\epsilon_2}{1 - \alpha_2}\right)\cos \frac{1}{2}\psi$$

$$-\frac{N\alpha_1}{\nu^2 + \beta^2}\left[\frac{\Delta(\sin \nu\psi, \sin \frac{1}{2}\psi)}{\nu}\int_{-\psi}^{\psi} G(\theta_1)\cos \nu\theta_1 \, d\theta_1\right.$$

$$\left. - \frac{\Delta(\sinh \beta\psi, \sin \frac{1}{2}\psi)}{\beta}\int_{-\psi}^{\psi} G(\theta_1)\cosh \beta\theta_1 \, d\theta_1\right]$$

and

$$2 \frac{\Delta(\sin \nu\psi, \cos \frac{1}{2}\psi)}{\alpha_2/4 - \nu^2} B_1 + 2 \frac{\Delta(\sinh \beta\psi, \cos \frac{1}{2}\psi)}{\alpha_2/4 + \beta^2} B_2$$

(16b)

$$= -\frac{N\alpha_1}{\nu^2 + \beta^2}\left[-\frac{\Delta(\cos \nu\psi, \cos \frac{1}{2}\psi)}{\nu}\int_{-\psi}^{\psi} G(\theta_1)\sin \nu\theta_1 \, d\theta_1\right.$$

$$\left. + \frac{\Delta(\cosh \beta\psi, \cos \frac{1}{2}\psi)}{\beta}\int_{-\psi}^{\psi} G(\theta_1)\sinh \beta\theta_1 \, d\theta_1\right].$$

The other necessary equations are found by applying the "radiation boundary condition" of (9) in the form $du/d\theta = 0$, $\theta = \pm\psi$. The derivative of the complementary solution is

(17a) $\quad \dfrac{du_c}{d\theta} = \nu(-A_1 \sin \nu\theta + B_1 \cos \nu\theta) + \beta(A_2 \sinh \beta\theta + B_2 \cosh \beta\theta),$

and the derivative of the particular solution is

$$\frac{du_1}{d\theta} =: -\frac{\alpha_1 N}{\nu^2 + \beta^2}\left[\frac{\nu^2 - \alpha_2/4}{2}\int_{-\psi}^{\psi} G(\theta_1)\cos \nu(\theta - \theta_1)\,\text{sgn}(\theta - \theta_1)\,d\theta_1\right.$$

(17b)

$$\left. + \frac{\beta^2 + \alpha_2/4}{2}\int_{-\psi}^{\psi} G(\theta_1)\cosh \beta(\theta - \theta_1)\,\text{sgn}(\theta - \theta_1)\,d\theta_1\right].$$

Now, since we wish to evaluate the derivative at $\theta = \pm\psi$, we note that sgn $(\psi - \theta_1) = +1$ and sgn $(-\psi - \theta_1) = -1$, so that the second pair of equations for A_i, B_i can be written:

$$-\nu \sin(\nu\psi)A_1 + \beta \sinh(\beta\psi)A_2 + \nu \cos(\nu\psi)B_1 + \beta \cosh(\beta\psi)B_2$$

(18a)

$$= \frac{N\alpha_1}{\nu^2 + \beta^2}\left[\frac{\nu^2 - \alpha_2/4}{2}\int_{-\psi}^{\psi} G(\theta_1)\cos \nu(\psi - \theta_1)\,d\theta_1\right.$$

$$\left. + \frac{\beta^2 + \alpha_2/4}{2}\int_{-\psi}^{\psi} G(\theta_1)\cosh \beta(\psi - \theta_1)\,d\theta_1\right];$$

and

$$(18b) \quad
\begin{aligned}
&\nu \sin(\nu\psi)A_1 - \beta \sinh(\beta\psi)A_2 + \nu \cos(\nu\psi)B_1 + \beta \cosh(\beta\psi)B_2 \\
&= -\frac{N\alpha_1}{\nu^2 + \beta^2}\left[\frac{\nu^2 - \alpha_2/4}{2} \int_{-\psi}^{\psi} G(\theta_1) \cos \nu(\psi + \theta_1)\, d\theta_1 \right. \\
&\left. \qquad + \frac{\beta^2 + \alpha_2/4}{2} \int_{-\psi}^{\psi} G(\theta_1) \cosh \beta(\psi + \theta_1)\, d\theta_1 \right].
\end{aligned}$$

It is convenient to add and subtract (18a, b) to get two separate sets of two equations each for the even components (A_1, A_2) of u_c and for the odd components (B_1, B_2). Thus we get

$$(19a) \quad
\begin{aligned}
&\nu \sin(\nu\psi)A_1 - \beta \sinh(\beta\psi)A_2 \\
&= -\frac{N\alpha_1}{\nu^2 + \beta^2}\left[\frac{\nu^2 - \alpha_2/4}{2} \int_{-\psi}^{\psi} G(\theta_1) \cos \nu\psi \cos \nu\theta_1\, d\theta_1 \right. \\
&\left. \qquad + \frac{\beta^2 + \alpha_2/4}{2} \int_{-\psi}^{\psi} G(\theta_1) \cosh \beta\psi \cosh \beta\theta_1\, d\theta_1 \right]
\end{aligned}$$

and

$$(19b) \quad
\begin{aligned}
&\nu \cos(\nu\psi)B_1 + \beta \cosh(\beta\psi)B_2 = -\frac{N\alpha_1}{\nu^2 + \beta^2}\left[-\frac{\nu^2 - \alpha_2/4}{2} \right. \\
&\left. \cdot \int_{-\psi}^{\psi} G(\theta_1) \sin \nu\psi \sin \nu\theta_1\, d\theta_1 + \frac{\beta^2 + \alpha_2/4}{2} \int_{-\psi}^{\psi} G(\theta_1) \sinh \beta\psi \sinh \beta\theta_1\, d\theta_1 \right].
\end{aligned}$$

Equations (16a, b) and (19a, b) suffice for the determination of the four constants A_1, A_2, B_1, B_2. There remains the matter of determination of the average temperature T_0. From (6b) we have $\int_{-\psi}^{\psi} u(\theta)\, d\theta = 0$. Substitution of the solution given by (14) yields the equation

$$(20) \quad
\begin{aligned}
0 &= 2\frac{\sin(\nu\psi)}{\nu} A_1 + 2\frac{\sinh(\beta\psi)}{\beta} A_2 - \frac{N\alpha_1}{\nu^2 + \beta^2}\left[-\frac{\alpha_2}{4}\frac{\nu^2 + \beta^2}{\nu^2 \beta^2} \right. \\
&\quad \cdot \int_{-\psi}^{\psi} G(\theta_1)\, d\theta_1 + \frac{\alpha_2/4 - \nu^2}{\nu^2} \int_{-\psi}^{\psi} F(\theta_1) \cos \nu\psi \cos \nu\theta_1\, d\theta_1 \\
&\left. \quad + \frac{\alpha_2/4 + \beta^2}{\beta^2} \int_{-\psi}^{\psi} F(\theta_1) \cosh \beta\psi \cosh \beta\theta_1\, d\theta_1 \right] + \frac{\psi}{2}\frac{\epsilon_2/\epsilon_1}{1 - \alpha_2}.
\end{aligned}$$

Since A_1, A_2 are now known, and $F(\theta) = f(\theta)/\sigma T_0^4$, (20) will give a relation from which T_0 can be found. This completes the derivation of the general solution. A summary of the formulae needed for the solution of the integro-differential equation (8) is given in the appendix.

4. Circular cylinder in uniform radiation field. We now specialize to the case of a complete closed arc $\psi = \pi$ and further to a uniform and parallel field of incident radiation. Then for the forcing function we have

$$(21) \qquad f(\theta) = e \begin{cases} \cos\theta & \text{if } |\theta| \leq \dfrac{\pi}{2}, \\[2mm] 0 & \text{if } |\theta| > \dfrac{\pi}{2}. \end{cases}$$

Since the input is symmetric in θ, we can expect that $B_1 = B_2 = 0$ in the solution. Equations (16a) and (19a) for determination of A_1, A_2 become

$$\frac{\nu \sin \nu\pi}{\alpha_2/4 - \nu^2} A_1 - \frac{\beta \sinh \beta\pi}{\alpha_2/4 + \beta^2} A_2$$

$$(22a) \qquad = \frac{N\alpha_1}{\nu^2 + \beta^2} \left[-\frac{1}{2\alpha_1}\left(\epsilon_1 + \frac{\epsilon_2}{1 - \alpha_2}\right)\left(\frac{\sin 2\nu\pi}{\nu} - \frac{\sinh 2\beta\pi}{\beta}\right) \right.$$

$$\left. + \frac{e}{\sigma T_0^4}\left(\frac{\cos \nu\pi \cos \dfrac{\nu}{2}\pi}{1 - \nu^2} - \frac{\cosh \beta\pi \cosh \dfrac{\beta}{2}\pi}{1 + \beta^2}\right) \right]$$

and

$$\nu \sin \nu\pi \, A_1 - \beta \sinh \beta\pi \, A_2$$

$$= \frac{N\alpha_1}{\nu^2 + \beta^2}\left\{ \frac{e}{\sigma T_0^4}\left[\left(\frac{\alpha_2}{4} - \nu^2\right)\frac{\cos \nu\pi \cos \dfrac{\nu}{2}\pi}{1 - \nu^2} - \left(\frac{\alpha_2}{4} + \beta^2\right)\right.\right.$$

$$(22b)$$

$$\left.\frac{\cosh \beta\pi \cosh \dfrac{\beta}{2}\pi}{1 + \beta^2}\right] - \frac{1}{2\alpha_1}\left(\epsilon_1 + \frac{\epsilon_2}{1 - \alpha_2}\right)\left[\left(\frac{\alpha_2}{4} - \nu^2\right)\frac{\sin 2\nu\pi}{\nu}\right.$$

$$\left.\left. - \left(\frac{\alpha_2}{4} + \beta^2\right)\frac{\sinh 2\beta\pi}{\beta}\right]\right\}.$$

The solution of this pair of equations for A_1 and A_2 yields

$$A_1 = \frac{N\alpha_1}{\nu^2 + \beta^2}\left(\frac{\alpha_2}{4} - \nu_2\right)\frac{\cot \nu\pi}{\nu}$$

$$(23a)$$

$$\cdot\left[\frac{e}{\sigma T_0^4}\frac{\cos \dfrac{\nu}{2}\pi}{1 - \nu^2} - \frac{1}{\alpha_1}\left(\epsilon_1 + \frac{\epsilon_2}{1 - \alpha_2}\right)\frac{\sin \nu\pi}{\nu}\right]$$

and

$$A_2 = \frac{N\alpha_1}{\nu^2 + \beta^2}\left(\frac{\alpha_2}{4} + \beta^2\right)\frac{\coth \beta\pi}{\beta}$$

(23b)

$$\cdot\left[\frac{e}{\sigma T_0{}^4}\frac{\cosh \frac{\beta}{2}\pi}{1 + \beta^2} - \frac{1}{\alpha_1}\left(\epsilon_1 + \frac{\epsilon_2}{1 - \alpha_2}\right)\frac{\sinh \beta\pi}{\beta}\right].$$

Equation (20) for the determination of T_0 gives

(24) $$\pi = \frac{\alpha_1}{\epsilon_1}\frac{e}{\sigma T_0{}^4} \quad \text{or} \quad \sigma T_0{}^4 = \left(\frac{\alpha_1}{\epsilon_1}\right)\frac{e}{\pi}.$$

It is interesting to note that this relation follows much more easily from an over-all power balance for the cylinder, considered to be at the uniform temperature T_0: power input $= e \cdot (2a) \cdot \alpha_1$ and power output $= \epsilon_1 \cdot \sigma T_0{}^4 \cdot 2\pi a$, so that, as before,

$$\sigma T_0{}^4 = \frac{\alpha_1}{\epsilon_1}\frac{e}{\pi}.$$

Equations (23a, b) for A_1 and A_2 serve to determine the portion $u_c(\theta)$ of the entire solution. We must next write the expression for the particular solution $u_p(\theta)$. Using the function $f(\theta)$ defined in (21), and substituting in (12), we find

$$u_p(\theta) = \frac{e}{\sigma T_0{}^4}\frac{\alpha_1 N}{\nu^2 + \beta^2}\left(\frac{\alpha_2/4 - \nu^2}{2\nu}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos\theta_1\sin\nu\mid\theta - \theta_1\mid d\theta_1\right.$$

$$\left. - \frac{\alpha_2/4 + \beta^2}{2\beta}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos\theta_1\sinh\beta\mid\theta - \theta_1\mid d\theta_1\right) - \frac{1}{4}$$

$$+ \frac{N\alpha_1}{\nu^2 + \beta^2}\frac{1}{\alpha_1}\left(\epsilon_1 + \frac{\epsilon_2}{1 - \alpha_2}\right)\left(\frac{\nu^2 - \alpha_2/4}{\nu^2}\cos\nu\pi\cos\nu\theta\right.$$

$$\left. + \frac{\alpha_2/4 + \beta^2}{\beta^2}\cosh\beta\pi\cosh\beta\theta\right).$$

The integrals appearing in this expression will differ accordingly as θ is less than $\pi/2$ or greater than $\pi/2$. The result is, considering θ to lie in $(0, \pi)$ only, because of the even symmetry,

$$
u_p = \frac{\pi}{\alpha_1/\epsilon_1} \frac{N\alpha_1}{\nu^2 + \beta^2} \left\{ \frac{\frac{\alpha_2}{4} - \nu^2}{2\nu} \frac{2}{1 - \nu^2} \left[\begin{array}{c} \sin\frac{\nu\pi}{2} \cos\nu\theta - \nu\cos\theta \\[2mm] \cos\frac{\nu\pi}{2} \sin\nu\theta \end{array} \right] \right.
$$

(25)
$$
\left. - \frac{\frac{\alpha_2}{4} + \beta^2}{2\beta} \frac{2}{1 + \beta^2} \left[\begin{array}{c} \sinh\frac{\beta\pi}{2} \cosh\beta\theta - \beta\cos\theta \\[2mm] \cosh\frac{\beta\pi}{2} \sinh\beta\theta \end{array} \right] \right\} - \frac{1}{4}
$$

$$
+ \frac{N\alpha_1}{\nu^2 + \beta^2} \frac{1}{\alpha_1} \left(\epsilon_1 + \frac{\epsilon_2}{1 - \alpha_2} \right) \left(\frac{\nu^2 - \alpha_2/4}{\nu^2} \cos\nu\pi \cos\nu\theta \right.
$$

$$
\left. + \frac{\alpha_2/4 + \beta^2}{\beta^2} \cosh\beta\pi \cosh\beta\theta \right).
$$

The upper set of results holds for $0 \leq \theta \leq \frac{\pi}{2}$ and the lower for $\frac{\pi}{2} < \theta \leq \pi$.

The two parts of the solution, $u_p(\theta)$ and $u_c(\theta)$, can be combined to give

$$
u(\theta) = \frac{\pi N\epsilon_1}{\nu^2 + \beta^2} \left\{ \frac{1}{\nu} \frac{\frac{\alpha_2}{4} - \nu^2}{1 - \nu^2} \left[\begin{array}{c} \dfrac{\cos\nu\theta}{2\sin(\nu\pi/2)} - \nu\cos\theta \\[3mm] \dfrac{\cos\nu(\pi - \theta)}{2\sin(\nu\pi/2)} \end{array} \right] \right.
$$

(26)
$$
\left. + \frac{1}{\beta} \frac{\frac{\alpha_2}{4} + \beta^2}{1 + \beta^2} \left[\begin{array}{c} \dfrac{\cosh\beta\theta}{2\sinh(\beta\pi/2)} + \beta\cos\theta \\[3mm] \dfrac{\cosh\beta(\pi - \theta)}{2\sinh(\beta\pi/2)} \end{array} \right] \right\} - \frac{1}{4},
$$

where once again the upper formulae hold for $0 \leq \theta \leq \frac{\pi}{2}$ and the lower for $\frac{\pi}{2} < \theta \leq \pi$. We note that the temperature difference between horizontally opposed points on the shell is particularly simple:

$$
(27a) \quad u(\theta) - u(\pi - \theta) = \frac{\pi N\epsilon_1}{\nu^2 + \beta^2} \left(\frac{\alpha_2/4 + \beta^2}{1 + \beta^2} - \frac{\alpha_2/4 - \nu^2}{1 - \nu^2} \right) \cos\theta
$$

$$
(27b) \qquad\qquad = \frac{\pi N\alpha_1}{\alpha_1/\epsilon_1} \frac{1}{1 + 4N\epsilon_1 + \dfrac{4N\epsilon_2}{1 - \alpha_2/4}} \cos\theta.
$$

The maximum temperature difference occurs when $\theta = 0$.

It can be seen that if the conductivity is neglected in the formulation of the present problem, then from (8) the problem reduces to that of solving an integral equation that is linear in $T^4(\theta)$. It is interesting to determine whether the solution to the more general problem, given by (26), reduces properly in the limit $k \to 0$ to the solution of the simpler problem. In this limit $N \to \infty$,

$$(28) \qquad \frac{\beta^2}{N} \to 4(\epsilon_1 + \epsilon_2), \quad \text{and} \quad \nu^2 \to \frac{1}{4} \frac{\epsilon_1 \, \alpha_2}{\epsilon_1 + \epsilon_2}.$$

Using these, together with the approximation (7), in (26) we find

$$
\begin{aligned}
\left(\frac{T}{T_0}\right)^4 &\doteq 1 + 4u \\[2mm]
(29) \qquad &= \frac{\pi \epsilon_1}{\epsilon_1 + \epsilon_2} \left[\begin{array}{c} \dfrac{1 - \alpha_2/4}{1 - \nu^2} \cos\theta + \dfrac{\alpha_2/4 - \nu^2}{1 - \nu^2} \dfrac{\cos \nu\theta}{2\nu \sin \dfrac{\nu}{2}\pi} \\[6mm] \dfrac{\alpha_2/4 - \nu^2}{1 - \nu^2} \dfrac{\cos \nu(\pi - \theta)}{2\nu \sin \dfrac{\nu}{2}\pi} \end{array} \right],
\end{aligned}
$$

which is just the answer obtained by solving the integral equation for $(T/T_0)^4$ with the above definition of ν^2.

A difference between the exact and approximate results will now arise from the different approximations of T/T_0 as $(1 + 4u)^{1/4}$ and $1 + u$. This circumstance suggests that for the low values of thermal conductivity, when $N \to \infty$, a new approximation might be useful. Thus let us "linearize" in terms of T^4, taking as a new variable $\tau = (T/T_0)^4$. The linearization now involves the approximation of

$$\frac{d^2}{d\theta^2}(T/T_0) = \frac{1}{4}(T/T_0)^{-3}\frac{d^2\tau}{d\theta^2} - 3(T/T_0)^{-1}\left[\frac{d}{d\theta}(T/T_0)\right]^2$$

by the single term $\dfrac{d^2}{d\theta^2}(T/T_0) \doteq \dfrac{1}{4}\dfrac{d^2\tau}{d\theta^2}$. Unfortunately, it seems difficult to justify this approximation since the second derivative $d^2\tau/d\theta^2$ vanishes in the range of interest, while the derivative $d(T/T_0)/d\theta$ does not. It might be noted, however, that in the event $N \to \infty$, which is the case when conductivity is low, the derivative term in (5) is divided by the large number N and so it may not matter greatly that the approximation is not as good as it might be.

If (5) is solved using the approximation $\dfrac{d^2}{d\theta^2}(T/T_0) \doteq \dfrac{1}{4}\dfrac{d^2\tau}{d\theta^2}$, where $\tau = (T/T_0)^4$, the result is $\tau(\theta) = (T/T_0)^4 = 1 + 4u(\theta)$, where $u(\theta)$ is

given in (26). The expression for temperature is now

(30) $$T/T_0 = (1 + 4u)^{1/4}.$$

The linearization used previously (see (7)) led to a truncated expansion of (30):

(30a) $$T/T_0 = 1 + u.$$

It is now suggested that formula (30) be used whenever conductivity is small. The value of conductivity k at which (30a) can be used may be decided by a comparison of the two results in any given set of cases with varying k. This matter will be illustrated numerically in the next section.

Next let us consider the case of very large conductivity k. The parameter $N \to 0$ and from the definitions of ν^2 and β^2 (see (10a)), we have $\nu^2 \to \alpha_2/4$ and $\beta^2 \to 4N\epsilon_1$. Hence for $k \to \infty$,

$$u(\theta) = \pi\epsilon_1 \lim_{N \to 0} \frac{N}{\beta} \left[\frac{\dfrac{\cosh \beta\theta}{2 \sinh \dfrac{\beta\pi}{2}} + \beta \cos \theta}{\dfrac{\cosh \beta(\pi - \theta)}{\sinh \dfrac{\beta\pi}{2}}} \right] - \frac{1}{4}$$

$$= \pi\epsilon_1 \lim_{N \to 0} \frac{\sqrt{N}}{2\sqrt{\epsilon_1}} \left(\frac{1}{2 \sinh \pi\sqrt{N\epsilon_1}} \right) - \frac{1}{4}$$

$$= 0.$$

This result is in agreement with physical reasoning, because material of infinitely high conductivity will have a uniform temperature, and this must be just T_0, the value derived above on the basis of simple equilibrium considerations.

5. Representative results. As data for obtaining some numerical results' the radiation field was chosen to correspond to that of the sun at about the distance of the earth therefrom. The value of the constant e in (21) is then the solar constant, which we take as $e = 0.033$ cal/cm^{2}sec. The dimensions of the cross section were chosen to be $a = 50$ cm and $t = 1$ cm. The value of the Stefan-Boltzmann constant is $\sigma = 1.39 \times 10^{-12}$ (cal/cm^{2}sec) $\cdot$ (°K)$^{-4}$.

There remain the absorptivities, emissivities, and the conductivity to be chosen. As an example, let us consider the effect of varying conductivity from a value of 0 to higher values, while keeping the radiation parameters ϵ_1, α_1, ϵ_2, α_2 fixed. Choose first $\epsilon_1 = 0.2$, $\alpha_1 = 0.5$, $\epsilon_2 = \alpha_2 = 1$. The distribution for $k = 0$ is found from (29), and we note that the slope of the

temperature distribution is discontinuous at $\theta = \dfrac{\pi}{2}$. A graph of this temperature variation with θ is shown in Fig. 3, where it can be seen that the point of discontinuity is also the point of minimum temperature. In this case of no heat transfer by conduction, the rear of the cylinder can only be heated by radiation from the interior of the forward portion, and it is clear that the top and bottom portions ($\theta = \pm\pi/2$) are least favorably situated in this respect. This applies, of course, in the present instance of an oncoming parallel field of radiation impinging upon the forward portion of the cylinder.

The ability of the material of the cylinder to conduct heat will ameliorate this effect, as can be seen in Fig. 3, where curves for positive values of k also appear. These curves were calculated with the modification suggested in (30). The slope of the temperature distribution is now continuous at $\theta = \dfrac{\pi}{2}$, and the minimum temperature no longer occurs at this position. In fact, increasing conductivity soon masks the entire effect, and for some k slightly greater than 0.1 (Fig. 3), the minimum temperature occurs at $\theta = 180°$. Higher values of k tend to decrease the total variation of temperature which approaches, as we have seen, the limit $T = T_0$ as $k \to \infty$.

It is of interest however to inquire somewhat more closely into the presence of a minimum temperature between $\theta = \dfrac{\pi}{2}$ and $\theta = \pi$. If we differentiate (26) and set the derivative to zero, we find that there is a minimum in $\dfrac{\pi}{2} < \theta < \pi$ if the following equation has a solution:

$$
(31) \quad \sin \nu(\pi - \theta) = \left(\frac{1 - \nu^2 \dfrac{\alpha_2}{4} + \beta^2}{1 + \beta^2 \dfrac{\alpha_2}{4} - \nu^2} \frac{\sin \dfrac{\nu\pi}{2}}{\sinh \dfrac{\beta\pi}{2}} \right) \sinh \beta(\pi - \theta)
$$

$$
= K \sinh \beta(\pi - \theta).
$$

Consider the behavior of each side of this equation near $\pi - \theta = 0$. The left side starts as $f_L(\theta) \doteq \nu(\pi - \theta)$ and the right as $f_R(\theta) = K\beta(\pi - \theta)$. From a sketch, one easily sees that because of the behavior of the sine and the hyperbolic sine there can be a solution only in the event that

$$
(32) \qquad\qquad\qquad \nu > K\beta.
$$

This is the criterion which determines whether conduction of heat smooths the distribution of temperature in the ring so that the minimum temperature occurs at the rearmost point $\theta = \pi$. Use of this criterion gives results in agreement with those obtained visually from Fig. 3; the critical value of
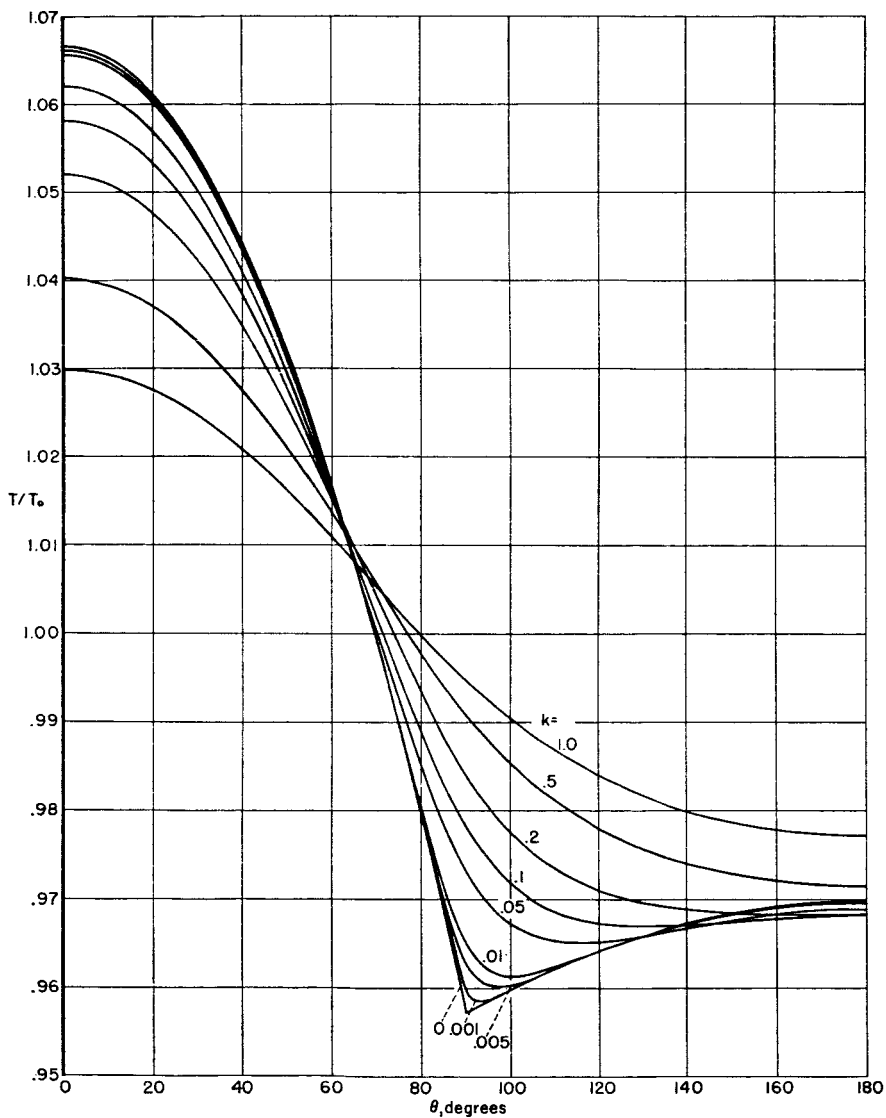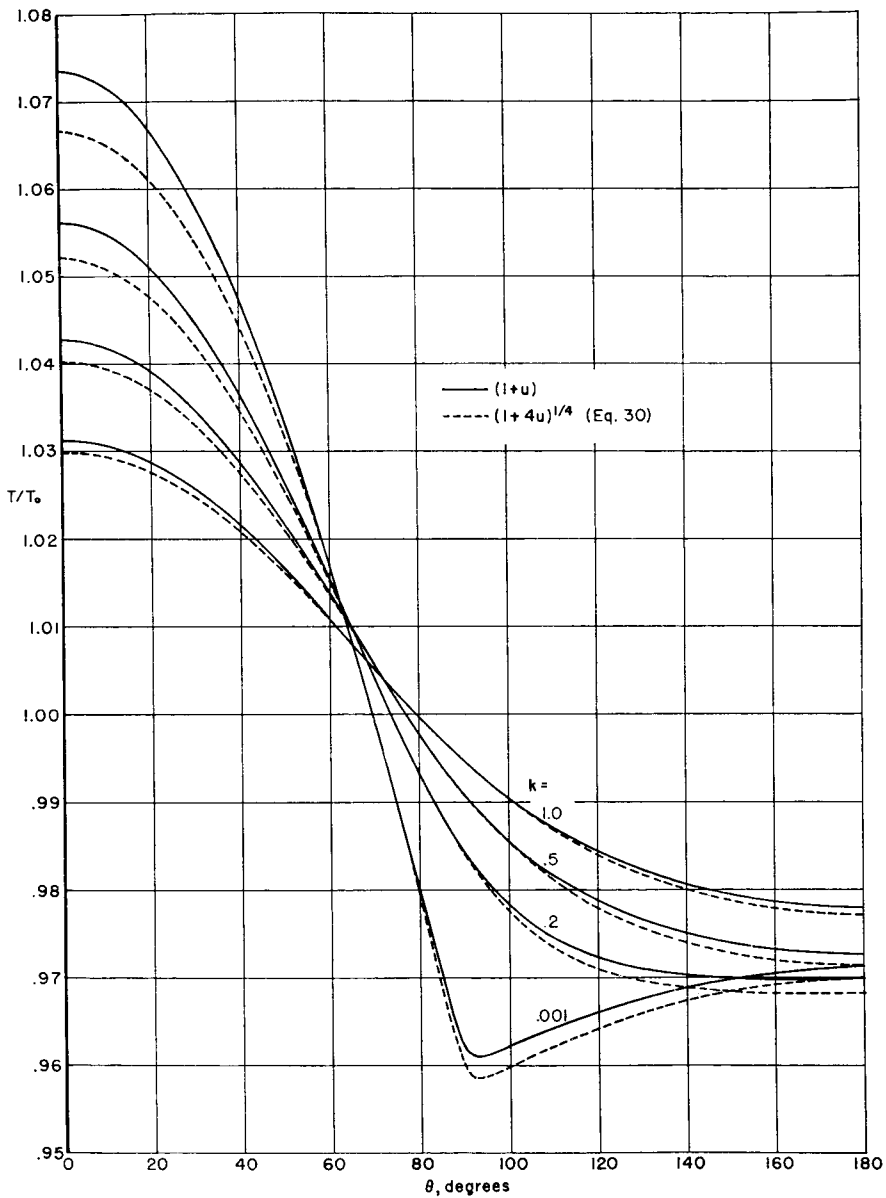
Fig. 3. *Variation of temperature distribution with conductivity k; $\alpha_1 = 0.5$, $\epsilon_1 = 0.2$, $\alpha_2 = \epsilon_2 = 1$.*

k, where the distribution ceases to have its minimum ahead of $\theta = \pi$, is slightly less than 0.2.

6. Accuracy of approximation as shown by numerical results. The results for T/T_0 for the cases studied above and calculated with T/T_0

FIG. 4. *Comparison of two approximations for T/T_0*

$= 1 + u$ are shown in Fig. 4. From these, the value of $u(\theta)$ can be read directly, and it is seen that even for the low conductivities its value is reasonably small (less than 0.07). It appears reasonable to accept such results and neglect the nonlinear terms. One should note, however, that this conclusion applies for a particular pair of values of ϵ_1, α_1. Higher values than the ones used here lead to larger values for $u(\theta)$, and each case should be examined on its own merit.

Some results from Fig. 3 are shown in Fig. 4 as dashed lines to give an idea of the range of applicability of the suggested improvement of (30). We see that the employment of this approximation improves the calculations considerably for low values of conductivity while making little difference for the high values. Thus the use of the fourth root approximation rather than the linear approximation seems to render the solution uniformly valid in k.

<div align="center">APPENDIX</div>

In this appendix we give in collected form the integro-differential equation, its solution, and the necessary auxiliary formulae.

The equation to be solved is

$$\text{(A1)} \quad \begin{aligned} &\frac{d^2u}{d\theta^2} - 4N(\epsilon_1 + \epsilon_2)u \\ &- \frac{1-\alpha_2}{4} \int_{-\psi}^{\psi} \left[\frac{d^2u}{d\theta^2} - 4N\left(\epsilon_1 + \frac{\epsilon_2}{1-\alpha}\right) u \right] \sin\frac{1}{2} |\theta - \theta_1| \, d\theta_1 \\ &= -N\alpha_1 \left[G(\theta) - \frac{1-\alpha_2}{4} \int_{-\psi}^{\psi} G(\theta_1) \sin\frac{1}{2} |\theta - \theta_1| \, d\theta_1 \right] - N\frac{\alpha_2 \epsilon_2}{1 - \alpha_2}. \end{aligned}$$

The boundary conditions are

$$\text{(A2)} \qquad \frac{du}{d\theta} = 0 \quad \text{at} \quad \theta = \pm\psi.$$

The solution is

$$\text{(A3)} \quad u(\theta) = u_p(\theta) + A_1 \cos \nu\theta + B_1 \sin \nu\theta + A_2 \cosh \beta\theta + B_2 \sinh \beta\theta,$$

where

$$\text{(A3a)} \quad \begin{aligned} u_p(\theta) = -\frac{\alpha_1 N}{\nu^2 + \beta^2} &\left[\frac{\nu^2 - \alpha_2/4}{2\nu} \int_{-\psi}^{\psi} G(\theta_1) \sin \nu \, |\theta - \theta_1| \, d\theta_1 \right. \\ &\left. + \frac{\beta^2 + \alpha_2/4}{2\beta} \int_{-\psi}^{\psi} G(\theta_1) \sinh \beta \, |\theta - \theta_1| \, d\theta_1 \right] + \frac{1}{4} \frac{\epsilon_2/\epsilon_1}{1 - \alpha_2}. \end{aligned}$$

The constants ν^2, β^2 are given by

(A3b)
$$\nu^2 = \tfrac{1}{2}\{\sqrt{[4N(\epsilon_1 + \epsilon_2) - \alpha_2/4]^2 + 4\epsilon_1\,\alpha_2\,N} - 4N(\epsilon_1 + \epsilon_2) + \alpha_2/4\},$$
$$\beta^2 = \nu^2 + 4N(\epsilon_1 + \epsilon_2) - \alpha_2/4.$$

For the boundary condition (A2), the equations for the determination of the constants A_i, B_i are:

(A4a)
$$\frac{\Delta(\cos \nu\psi, \sin \tfrac{1}{2}\psi)}{\alpha_2/4 - \nu^2}\,A_1 + \frac{\Delta(\cosh \beta\psi, \sin \tfrac{1}{2}\psi)}{\alpha_2/4 + \beta^2}\,A_2$$
$$= -\frac{1}{2\alpha_2\,\epsilon_1}\left(\epsilon_1 + \frac{\epsilon_2}{1 - \alpha_2}\right)\cos \frac{1}{2}\,\psi - \frac{N\alpha_1}{\nu^2 + \beta^2}\left[\frac{\Delta(\sin \nu\psi, \sin \tfrac{1}{2}\psi)}{2\nu}\right.$$
$$\left. \cdot \int_{-\psi}^{\psi} G(\theta_1)\,\cos \nu\theta_1\,d\theta_1 - \frac{\Delta(\sinh \beta\psi, \sin \tfrac{1}{2}\psi)}{2\beta}\int_{-\psi}^{\psi} G(\theta_1)\,\cosh \beta\theta_1\,d\theta_1\right];$$
$$\nu \sin \nu\psi\,A_1 - \beta \sinh \beta\psi\,A_2 = -\frac{N\alpha_1}{\nu^2 + \beta^2}\left[\frac{\nu^2 - \alpha_2/4}{2}\right.$$
$$\left. \cdot \int_{-\psi}^{\psi} G(\theta_1)\,\cos \nu\psi\,\cos \nu\theta_1\,d\theta_1 + \frac{\beta^2 + \alpha_2/4}{2}\int_{-\psi}^{\psi} G(\theta_1)\,\cosh \beta\psi\,\cosh \beta\theta_1\,d\theta_1\right];$$

and

(A4b)
$$\frac{\Delta(\sin \nu\psi, \cos \tfrac{1}{2}\psi)}{\alpha_2/4 - \nu^2}\,B_1 + \frac{\Delta(\sinh \beta\psi, \cos \tfrac{1}{2}\psi)}{\alpha_2/4 + \beta^2}\,B_2$$
$$= -\frac{N\alpha_1}{\nu^2 + \beta^2}\left[\frac{\Delta(\cos \nu\psi, \cos \tfrac{1}{2}\psi)}{2\nu}\int_{-\psi}^{\psi} G(\theta_1)\,\sin \nu\theta_1\,d\theta_1\right.$$
$$\left. - \frac{\Delta(\cosh \beta\psi, \cos \tfrac{1}{2}\psi)}{2\beta}\int_{-\psi}^{\psi} G(\theta_1)\,\sinh \beta\theta_1\,d\theta_1\right];$$
$$\nu \cos \nu\psi\,B_1 + \beta \cosh \beta\psi\,B_2 = -\frac{N\alpha_1}{\nu^2 + \beta^2}\left[\frac{\alpha_2/4 - \nu^2}{2}\right.$$
$$\left. \cdot \int_{-\psi}^{\psi} G(\theta_1)\,\sin \nu\psi\,\sin \nu\theta_1\,d\theta_1 + \frac{\alpha_2/4 + \beta^2}{2}\int_{-\psi}^{\psi} G(\theta_1)\,\sinh \beta\psi\,\sinh \beta\theta_1\,d\theta_1\right].$$

Note that $\Delta(f(\psi), g(\psi)) = \begin{vmatrix} f & g \\ f' & g' \end{vmatrix}$, where the primes denote derivatives with respect to ψ.

If boundary conditions different from (A2) are desired, the requisite modifications are made to the second equation in each of the sets (A4a) and (A4b). The first equation in each set is a consequence of the method of solution (by reduction to an ordinary differential equation of higher order) and therefore does not depend upon the boundary conditions.

Some of the integrals which arise in the calculation of this solution are listed next. They are of interest in themselves, in addition to their utility in the solution of integral equations with kernels like that of (A1). In these integrals, the variables θ, θ_2 lie in the interval $(-\psi, \psi)$

$$\int_{-\psi}^{\psi} \cos \alpha\theta_1 \sin \nu \mid \theta - \theta_1 \mid d\theta_1 = \frac{2}{\nu^2 - \alpha^2} \left[\nu \cos \alpha\theta - \Delta(\cos \alpha\psi, \sin \nu\psi) \cos \nu\theta \right].$$

$$\int_{-\psi}^{\psi} \cos \alpha\theta_1 \sinh \beta \mid \theta - \theta_1 \mid d\theta_1$$

$$= \frac{2}{\beta^2 + \alpha^2} \left[-\beta \cos \alpha\theta + \Delta(\cos \alpha\psi, \sinh \beta\psi) \cosh \beta\theta \right].$$

$$\int_{-\psi}^{\psi} \sin \alpha\theta_1 \sin \nu \mid \theta - \theta_1 \mid d\theta_1$$

$$= \frac{2}{\nu^2 - \alpha^2} \left[\nu \sin \alpha\theta + \Delta(\sin \alpha\psi, \cos \nu\psi) \sin \nu\theta \right].$$

$$\int_{-\psi}^{\psi} \sin \alpha \mid \theta_2 - \theta_1 \mid \sin \nu \mid \theta - \theta_1 \mid d\theta_1 = \frac{2}{\nu^2 - \alpha^2} \left[\nu \sin \alpha \mid \theta - \theta_2 \mid \right.$$

$$- \alpha \sin \nu \mid \theta - \theta_2 \mid + \Delta(\sin \nu\psi, \sin \alpha\psi) \cos \nu\theta \cos \alpha\theta_2$$

$$\left. + \Delta(\cos \nu\psi, \cos \alpha\psi) \sin \nu\theta \sin \alpha\theta_2 \right].$$

Further combinations involving hyperbolic functions can be found by giving imaginary values to the parameters ν and α; for example, the second formula above follows from the first by putting $\nu = i\beta$.

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